# RATIONAL AND IRRATIONAL NUMBERS
## (NÚMEROS RACIONAIS E IRRACIONAIS)

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### ABSTRACT

A theory of representation of both the rational and irrational numbers in the interval [0,1) by a balanced binary tree is developed, such that it is possible to contradict Cantor's theory that it is not possible to enumerate the Real numbers; provide a calculation of the functional limits of the quantities of rational and irrational numbers in the interval; and conclude that the irrational numbers, according to this theory, are just unattainable theoretical limits of infinitely precise approximations by rational numbers.

***K*eywords** Cantor · Irrational Numbers · Rational Numbers · Enumeration · Theory

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**Conflict of Interest**

*The author of this manuscript affirms that there are no conflicts of interest to disclose regarding the publication of this article. The author has no financial or personal relationships that could inappropriately influence or bias the work reported in this manuscript.*

## 1 Introduction

In this study, I develop a theory of the rational and irrational numbers and their representation by a balanced binary tree whose notes represent increasingly precise subdivisions of the interval [0,1). This representation permits the proof of the theory and the calculation of the limits of the quantities of both rational and irrational numbers in this interval. The theory also allows for an enumeration of all the Real numbers in the interval [0,1) and, also, in the entire Real line, contradicting the theory of Cantor that such enumeration would be impossible. The developed theory also asserts that the irrational numbers are just unattainable theoretical limits of infinitely precise approximations by rational numbers. I refer the reader to the previous article [8] for a better understanding of the development of the theory.

## 2 Methodology

In the present study, I use the same methodology used in [8], which I repeat here: traditional Mathematics methods and some Philosophy methods. According to [7] and [4], the methods of Philosophy are "the methods of reasoning and analysis that seek to clearly define the concepts used, investigate and expose the foundations of ideas and theories and build a systematic theory that is based on other ideas and systems of thought".

[0]Laerte Ferreira Morgado, Dr.BA (California Southern Universty/USA), M.Sc. Electrical Engineering (Unicamp/BR), Computer Engineer (Unicamp/BR), PgDip. Computer and Communications Law (University of London/UK), LL.B. (UnB/BR). Public Finance Consultant of the Brazilian Federal Senate since 2009. *As mentioned in [8], this work was developed entirely by myself, although I found some partially overlapping results on the Internet after I have developed the theory.*

In this context, it is essential to precisely define concepts, critique ideas, and compare them, emphasizing their limitations or strengths [4, 177]. Similarly, strive to uncover and elucidate the underlying principles of ideas, thereby laying the groundwork for a thorough critique. Essential to this study is the development of an integrated system that provides explanatory and insightful value. Hence, this study aligns with the positivist school of thought, employing "logic, reason, rigor, and inference in the pursuit of knowledge" [4, 189]. As per [7], "it also incorporates both deduction and induction, thereby drawing both 'necessary' and 'probable' conclusions [4, 190], aiming to both challenge an existing theory and formulate a new one."

## 3 Results and Discussion

As mentioned in [8, 9], consider the binary tree representation of the unlimited subdivision of the half-open interval on the Real line $[0, 1)$ as depicted in Figure 3. It is easily seen that the number of leaf nodes is $2^h$ for each height $h$ of the tree and the size of each of the sub-intervals represented by the leaves is $2^{-h}$.

(1) The union of the sub-intervals represented by the leaf nodes at height $h$ is $\bigcup_{i=1}^{2^h} S_i = [0, 1)$, which can be proved by mathematical induction, as the process of generation of the leafs for height $h$ from height $h-1$ just splits in half the previous interval, with no exclusion of points, and the root node ($h = 0$) represents the entire interval $[0, 1)$.

(2) The quantity of Real numbers is each of the leaf nodes $Q_i(S_i) \to 1$ as $h \to \infty$. In fact, each of the leaf nodes at height $h$, $S_i$, has at least one Real number, since all of them are left-closed. Although their sizes ($2^{-h}$) tend to 0 as $h \to \infty$, the open right end, despite tending to the left end, never "erases" it, i.e., the left ends of all the intervals represented by the leaf nodes at any height $h$ remain, as $h \to \infty$. In addition, if there were at least two $(2)$ points in any of the sub-intervals $S_i$ as $h \to \infty$, this would be a contradiction as the distances between these two points would necessarily be a positive number $d$, but, as the size of the interval $(2^{-h})$ tends to 0 as $h \to \infty$, at least one of these numbers would fall out of the interval for a sufficiently large $h$.

We can conclude that, in the limit ($h \to \infty$), the union of the leaves of the tree represents all Real numbers in the interval $[0, 1)$, and each of the leaves correspond to just one $(1)$ Real number. As already mentioned in [8, 9], we can enumerate all the nodes of this binary tree if we begin with the root node, attributing to it number 1, and, proceeding downwards to each of the following levels of the tree, in a zigzag manner, attribute to the nodes at height $h = 1$ numbers $\{2, 3\}$, to the nodes at height $h = 2$ numbers $\{4, 5, 6, 7\}$, and so on. In this way, we can enumerate all the Real numbers in the interval $[0, 1)$. With this method, we could also enumerate all the Real numbers in an arbitrary interval $(-x, x)$, $x \in \mathbb{R}$, by building an equivalent binary tree beginning with a root node representing the whole interval $(-x, x)$. By making this $x \to \infty$, we could enumerate all the Real numbers in a arbitrarily large interval of the Real line. This seems to contradict the famous result of Cantor that it would be impossible to enumerate the Real numbers.

As already proved in [8, 9], all the left and right ends of the sub-intervals represented by the nodes, and the leaf nodes especially, of the tree represent rational numbers expressed in binary notation with $h$ binary digits after the point $(.)$, for any height $h$. As we make $h \to \infty$, the union of these extremes of the sub-intervals of the leaves of the tree represents all the rational numbers in $[0, 1]$ and grows exponentially in size $(2^h)$, as $h \to \infty$, filling the interval $[0, 1]$.

For any height $h$ of the tree, and excluding the left and right ends of the sub-intervals represented by the leaves of the tree, which are all rational numbers, the interior of any of these intervals may be represented by $S_i = (l_i, r_i)$, where $l_i$ and $r_i$ are rational numbers, and their union by $\bigcup_{i=1}^{2^h} S_i = S_1 \cup S_2 \cup ... \cup S_{2^h} = (l_1, r_1) \cup (l_2, r_2) \cup ... \cup (l_{2^h}, r_{2^h})$.

Although many rational numbers may be inside any of the sub-intervals of $\bigcup_{i=1}^{2^h} S_i$ for any height $h$, as the extremes of these sub-intervals are all rational numbers, we can conclude that all the irrational numbers in $[0, 1)$ are contained in these sub-intervals, i.e. $\bigcup_{i=1}^{2^h} S_i$, for any height $h \to \infty$.

Consider, now, any supposedly irrational number $i \in [0, 1)$, at the root node of the tree. As we proceed building the tree for increasing values of $h \to \infty$, it can be easily be shown that $i$ belongs to an infinite series of nested sub-intervals down the tree $\{(l_1 = 0, r_1 = 1), (l_2, r_2), (l_3, r_3)...\}$, with $(0 = l_1) \leq l_2 \leq l_3 \leq ... < i < ... \leq r_3 \leq r_2 \leq (r_1 = 1)$, whose sizes tend to 0 as $h \to \infty$. For any irrational number $i$ and any height $h$ of the tree, the innermost sub-interval $(l_{(h,i)}, r_{(h,i)})$ represents a leaf node, such that $l_{(h,i)} < i < r_{(h,i)}$. It is easily shown that $\mid r_{(h,i)} - l_{(h,i)} \mid > 0$, for any $h$, and $\mid r_{(h,i)} - l_{(h,i)} \mid \to 0$ as $h \to \infty$. Therefore $l_{(h,i)} \to r_{(h,i)}$ and $r_{(h,i)} \to l_{(h,i)}$ as $h \to \infty$. In the same way, $l_{(h,i)} \to i$ and $r_{(h,i)} \to i$, as $h \to \infty$.

We can conclude that $l_{(h,i)}$ and $r_{(h,i)}$ are increasingly precise rational approximations for the irrational number $i$. At any height $h$ of construction of the tree, any irrational number $i$ necessarily belongs to an open interval corresponding to a leaf, so that the quantity of the elements of the union of all such $is$ is $\#\{\bigcup_i\} \geq 2^h$, where $2^h$ is the number of leaf nodes at height $h$.

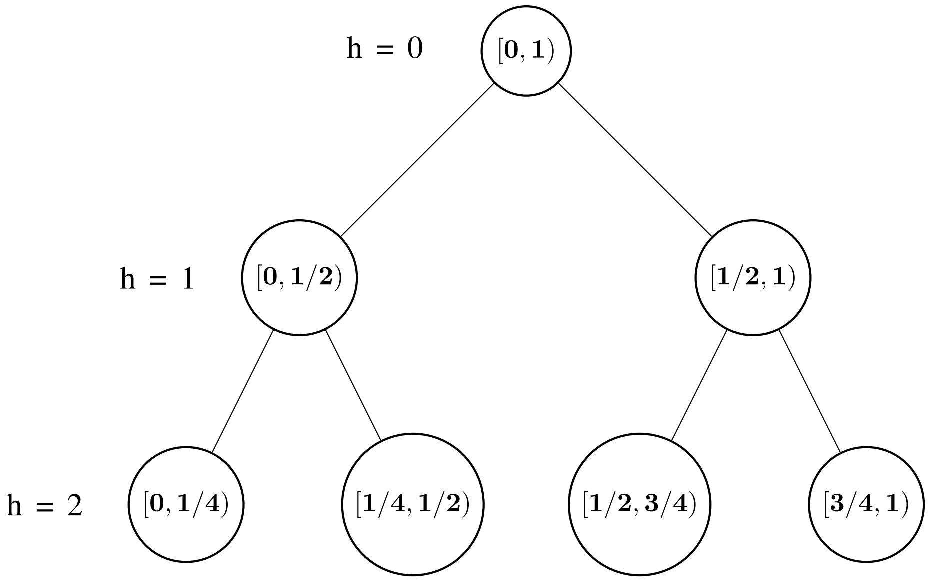


Figure 1: Balanced Binary Tree. Infinite Subdivision of the Interval [0,1]

However, as $h \to \infty$, there can be at most one $(1)$ such irrational number $i$ inside each leaf node, for, if there were two of them $i_1$ and $i_2$, $\mid i_2 - i_1 \mid >= d > 0$, which is contradictory to the fact that the size of the sub-interval represented by the leaf node $2^{-h} \to 0$ as $h \to \infty$.

The quantity of all the irrational numbers $i \in [0, 1)$, $\#\{\bigcup_i\} \leq 2^h$, therefore, as $h \to \infty$. So $\lim_{h\to\infty} \#\{\bigcup_i\} = \lim_{h\to\infty} 2^h$.

As the limit $h \to \infty$ is never reached, the irrational numbers $i$ will be approximated by rational numbers with infinitely increasing precision, such that, as $h \to \infty$, the quantity of rational numbers in each leaf node $QRac \to 1$ and the quantity of irrational numbers in each leaf node $QIrrac \to 1$, as $h \to \infty$.

That is, $\lim_{h\to\infty} QRac[0, 1) = \lim_{h\to\infty} QIrrac\,[0, 1) = \lim_{h\to\infty} 2^h$. But, as the quantity of Real numbers $QReal[0, 1) = \lim_{h\to\infty} 2^h$, as the number of Real numbers in each leaf of the tree approaches $(1)$ as $h \to \infty$, as demonstrated, in essence, **"the irrational numbers are just the unattainable theoretical limits of increasingly precise approximations of rational numbers"**. They are not representable, in binary notation, by any finite sequence of digits.

The irrational numbers can be approximated, as shown, by a pair of rational numbers with infinitely increasing precision; and the rational numbers become infinitely more precise, with infinitely increasing quantity of binary digits after the point "(.)", without ceasing to be rational numbers.

We could say that the irrational numbers are the "theoretical limits" of the rational numbers with increasing quantities of binary digits after the point "(.)".

In the practical technique of measurement in the physical-material world, there would no be irrational numbers, but only approximations by rational numbers with different degrees of precision. In Geometry, however, which is a theoretical science, the irrational numbers do exist, as is the case of the number $\sqrt{2}$, the size of the diagonal of a square with sides of length $1$. It is impossible to measure exactly this diagonal in practice, but the theoretical limit, in Geometry, can be calculated, and is the irrational number $\sqrt{2}$. Likewise, in any physical representation of the Real line, the approximations of rational numbers with increasing precision would leave no wholes to be filled, but gaps, whose lengths would tend to $0$. In a geometrical Real line, however, there would also be such gaps whose lengths tend to $0$, but, as there is no limit (or cap) to the precision that can be attained in the theory of Geometry, we could imagine pairs of rational numbers, as in the leaves of the depicted tree, tending to each other with infinite precision, but never reaching one another, so that, in the limit, there would exist a point between any such pair of rational numbers (and only one, as they tend to each other), representing an irrational number, that would surely exist.

## 4 Final Considerations

In the present work, based on a previous article [8], I develop a theory of the representation of both rational and irrational numbers by a balanced binary tree with infinitely increasing height and whose nodes represent subdivisions of the Real interval $[0, 1)$. This theory allows for the enumeration fo all the Real numbers in the interval $[0, 1)$, as well as in all the Real line, therefore potentially refuting Cantor's theory that such an enumeration would be impossible. It also allows for the calculation of the functional limits of the quantities of rational and irrational numbers in this interval. It concludes that the irrational numbers are just unattainable theoretical limits of infinitely precise approximations by rational numbers. They would exist in Geometry and in the geometrical Real line, but not attainable in the material physical world.

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